Correlation functions in the non-commutative Wess-Zumino-Witten model [1]

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Abstract

We develop a systematic perturbative expansion and compute the one-loop two-points, three-points and four-points correlation functions in a non-commutative version of the $U(N)$ Wess-Zumino-Witten model in different regimes of the θ-parameter showing in the first case a kind of phase transition around the value $\theta_c = \frac{\sqrt{p^2 + 4\,m^2}}{\Lambda^2\,p}$, where Λ is a ultraviolet cut-off in a Schwinger regularization scheme. As a by-product we obtain the functions of the renormalization group, showing they are essentially the same as in the commutative case but applied to the whole $U(N)$ fields; in particular there exists a critical point where they are null, in agreement with a recent background field computation of the beta-function, and the anomalous dimension of the Lie algebra-valued field operator agrees with the current algebra prediction. The non-renormalization of the level k is explicitly verified from the four-points correlator, where a left-right non-invariant counter-term is needed to render finite the theory, that it is however null on-shell. These results give support to the equivalence of this model with the commutative one.

1 Generalities

Non commutative (NC) field theories have recently attracted attention because of its comparison as effective theories in the context of D-brane physics [1]. Due to this fact mostly studied were NC gauge theories [2], [3] as well as toy models of scalar non derivative field theories [4], [5]. They are generally formulated as extensions of ordinary field theories where the usual point-to point product is replaced by the "*" product; only in two dimensions this procedure preserves Lorentz (or $SO(2)$ in euclidean formulations)

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invariance, however it necessary breaks (if there were) scale invariance explicitly. Being two dimensional conformal field theories (CFT) a subject vastly studied during the last two decades due to its implications in Statistical Mechanics and Solid State Physics other than in the perturbative formulation of Superstring Theories, a question naturally arise: do the NC extensions of two dimensional CFT define at quantum level another CFT? And if so, what? In the case of free field theories (free bosons and fermions) the answer is yes, but in a trivial way because both theories indeed coincide explicitely. And among the interacting theories more or less tractable by current algebra methods are the Wess-Zumino-Witten (WZW) models [6]. It is the aim of this paper to study perturbative aspects of these peculiar models that include infinite interaction vertices containing derivatives. For conventions adopted about NC spaces, groups definitions, etc., we refer the reader to Appendix A.

For definiteness we consider the non-commutative version of the $U(N)$ WZW model in euclidean space defined by the bare action

$$
\begin{aligned}
S[g] &= \frac{1}{\lambda_0{}^2}\left(I_0[g] + I_{WZ}[\tilde{g}]\right) \\
I_0[g] &= \frac{1}{2}\int_\Sigma d^2\vec{x}\, Tr\,(L_i(g)L_i(g)) \\
I_{WZ}[\tilde{g}] &= i\,\frac{\alpha_0}{3}\int_B Tr\,(L(\tilde{g})\wedge L(\tilde{g})\wedge L(\tilde{g}))
\end{aligned}
\tag{1.1}
$$

where λ_0 is the coupling constant of the theory and α_0 will be eventually identified with $\frac{\lambda_0^2\,k_0}{4\pi}$ with k_0 a parameter not to be renormalized. [3] The three-dimensional manifold "B" is taken to be a cylinder with the top and bottom disks of infinite radius (two $\Re^2$'s) parametrized by $\vec{x} = (x^1, x^2)$ while the height variable $s \in \Re$. The boundary conditions on $\tilde{g}(\vec{x}, s)$ are

$$
\begin{aligned}
\tilde{g}(\vec{x}, s) &\overset{x\to\infty}{\longrightarrow} 1 \\
\tilde{g}(\vec{x}, s) &\overset{s\to-\infty}{\longrightarrow} 1 \\
\tilde{g}(\vec{x}, s) &\overset{s\to\infty}{\longrightarrow} g(\vec{x})
\end{aligned}
\tag{1.2}
$$

Unless specified the contrary all the field products along the paper are understood as "$*$" products with parameter $\theta^{ij} \equiv \theta\,\epsilon^{ij}$, $\theta^{is} = 0$, where the matrix $\epsilon = i\,\sigma_2$ as usual in two dimensions and $\theta > 0$. Under these conditions usual properties of the commutative case hold and the effective degrees of freedom are represented by $g(\vec{x})$; in particular the Polyakov-Wiegmann formula

$$
\begin{aligned}
S[g\,h] &= S[g] + S[h] + \frac{1}{\lambda^2}\int d^2\vec{x}\, Tr\left(P_-^{ij}\, L_i(g)\, R_j(h)\right) \\
P_\pm^{ij} &\equiv \delta^{ij} \pm i\,\alpha_0\,\epsilon^{ij}
\end{aligned}
\tag{1.3}
$$

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[3] This fact is enforced in the commutative case where by topological reasons it must be an integer; here we will just verify that this property continues to hold in the NC case at least perturbatively, however the integer character is certainly not yielded by our analysis, see [7] in relation to this topic.

as well as cyclic properties under traces are valid. The first order variation of the action is ($z = x^1 + i\,x^2$)

$$
\begin{aligned}
\delta S[g] &\equiv S[g + \delta g] - S[g] = -\frac{1}{\lambda_0{}^2}\int d^2\vec{x}\; P_+^{ij}\, Tr\left(g^{-1}\,\delta g\;\partial_i\, L_j(g)\right)\\
&= \frac{1}{\lambda_0{}^2}\int i\,d\overline{z}\wedge dz\; Tr\left(g^{-1}\,\delta g\;\left(2\,g^{-1}\,\partial_z\partial_{\overline{z}}g + (1 + \alpha_0)\,\partial_z g^{-1}\,\partial_{\overline{z}}g\right.\right.\\
&\quad + \left.\left.(1 - \alpha_0)\,\partial_{\overline{z}}g^{-1}\,\partial_z g\right)\right)
\end{aligned}
\tag{1.4}
$$

In the critical case where (for definiteness) $\alpha_0 = 1$ yields as equations of motion the usual conservation of the currents $J \equiv \frac{k}{4\pi}\,R_z$, $\overline{J} \equiv -\frac{k}{4\pi}\,L_{\overline{z}}$

$$
\partial_{\overline{z}}\,J = \partial_z\,\overline{J} = 0
\tag{1.5}
$$

The action (1.1) is invariant under global transformations $g \rightarrow h_L g h_R$, with $h_L(h_R)$ belonging to any subgroup $H_R(H_L)$ of $U(N)$ isomorphic to $U(p), p \leq N$,. In the commutative critical case conformal invariance raises this invariance to holomorphic and anti-holomorphic dependence, being the generators of these transformations the momenta of the currents $J, \overline{J}$ that satisfy the standard level k left and right Kac-Moody algebra. In the NC case we have not certainly conformal invariance due to the introduction of the "*" product; however it seems that the holomorphic character of the currents (now defined with the "*" product) continues to hold. However we are not in conditions of asserting that they verify a current algebra because we have not to our disposal a hamiltonian formulation which yielded a canonical quantization of the theory, the main obstacle being the infinite time derivatives in the lagrangean that give a non-local character to the theory. [4] Then the existence or not of a conformal structure is in our opinion an open question, being one of the goals of the present work to shield some light about it.

2 The perturbative expansion

In reference [9] a computation of the one-loop beta function was made by using the background fiel method. It is known however that this method to compute the OPI effective action $\Gamma[\Phi]$ is not convenient beyond one-loop; from its path integral expression

$$
\begin{aligned}
\Delta\Gamma[\Phi] &\equiv \Gamma[\Phi] - S[\Phi] = -\ln\int D\xi\; e^{-S_q[\xi;\Phi] + \int d^d\vec{x}\,\frac{\delta\Delta\Gamma[\Phi]}{\delta\Phi}\,\xi}\\
S_q[\xi;\Phi] &\equiv S[\Phi + \xi] - S[\Phi] - \int d^d\vec{x}\,\frac{\delta S[\Phi]}{\delta\Phi}\,\xi
\end{aligned}
\tag{2.1}
$$

we see that at one-loop the source term is not present and the computation is straight, but beyond one-loop it must be included in the form of $\vec{x}$-dependent one-point vertices, the

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[4] We have no clear at all that the method advocated in reference [6] based in a Poisson structure determined by equation (1.4) led to a right answer in the NC context due to the presence of the "*" product, other the fact we do not know of generalizations of it. Recent work in reference [8] could help to address this question that certainly deserves further study.

procedure becoming more and more involved. This drawback is common to any QFT but in theories with fields living on group manifolds (as the present case) another one is present and has to do with the fact that if we write the group-valued field $g = \exp(\phi)$ with ϕ the fields of the theory living in the Lie algebra then the splitting corresponding to (2.1) in a background and a quantum part $\phi = \Phi + \xi$ identifying $g_0 \equiv \exp(\Phi)$ as the background field living in the group yields an awful expansion; the best suited splitting used in references [9], [10] is $g = g_0 \, \exp(\pi)$ but in order to be used beyond one-loop through (2.1) we would need the NC Baker-Campbell-Haussdorf formula to get $\xi = \xi[\pi; g_0]$ through the equality

$$g = \exp(\Phi + \xi) \leftrightarrow \exp(\pi) = \exp(-\Phi) \, \exp(\Phi + \xi) \tag{2.2}$$

and we must face another hard problem. So we prefer to develop a systematic perturbative series in the standard way.

To regulate in the infrared we add to (1.1) a mass term [10]

$$S_m[g] = \frac{m_0^2}{2\,\lambda_0^2} \int d^2\vec{x} \, Tr\left(g + g^{-1} - 2\right) \tag{2.3}$$

As showed in [10] dimensional regularization could be consistently used to regulate in the ultraviolet; however we need to add extra-dimensions terms, a new coupling, etc; to evite these drawbacks we adopt a Schwinger-like prescription [11] and take the free propagator

$$\frac{1}{k^2 + m_0{}^2} \longrightarrow \tilde{G}(k^2) = \frac{e^{-\frac{k^2}{2\Lambda^2}}}{k^2 + m_0{}^2} \tag{2.4}$$

where Λ is the UV cut-off.

The group element is parametrized as $g \equiv \exp\left(\lambda_0 \pi_0\right)$ where $\pi_0(\vec{x}) = \pi_0^a(\vec{x}) \, X_a$ is the quantum field living on $u(N)$ and obeying from (1.2) the boundary condition $\pi_0(\vec{x}) \overset{x \to \infty}{\longrightarrow} 0$. The action is then written

$$S[g] = S_q[\pi_0] + \sum_{n \geq 3} S^n[\pi_0]$$

$$S_q[\pi_0] = \frac{1}{2} \int \frac{d^2\vec{k}}{(2\,\pi)^2} \, \kappa_{b_1 b_2} \, \tilde{G}(k)^{-1} \, \tilde{\pi}_0^{b_1}(\vec{k}) \, \tilde{\pi}_0^{b_2}(-\vec{k}) \tag{2.5}$$

with $S^n[\pi_0]$ the vertices given in Appendix A. Here we quote explicitly in momentum space the first four ones including S_m. By denoting $\vec{k} \times \vec{p} \equiv \epsilon^{ij} k_i \, p_j$ we have

$$S^3[\pi_0] = \prod_{l=1}^{3} \left(\int \frac{d^2\vec{k}_l}{(2\pi)^2} \, \tilde{\pi}_0^{b_l}(\vec{k}_l) \right) (2\pi)^2 \, \delta^2 \left(\sum_{l=1}^{3} \vec{k}_l \right) \frac{\alpha_0 \, \lambda_0}{i\,9} \, \kappa_{b_1 b_2 b_3} \, v_c^3(\vec{k}_1, \vec{k}_2, \vec{k}_3)$$

$$v_c^3(\vec{k}_1, \vec{k}_2, \vec{k}_3) = \exp\left(\frac{\theta}{2i} \, \vec{k}_1 \times \vec{k}_2 \right) \vec{k}_1 \times \vec{k}_2 + (cyclic) \tag{2.6}$$

$$S^4[\pi_0] = \prod_{l=1}^{4} \left(\int \frac{d^2\vec{k}_l}{(2\pi)^2} \, \tilde{\pi}_0^{b_l}(\vec{k}_l) \right) (2\pi)^2 \, \delta^2 \left(\sum_{l=1}^{4} \vec{k}_l \right) \frac{\lambda_0^2}{4!\,2} \, \kappa_{b_1 \ldots b_4} \, v_c^4(\vec{k}_1, \ldots, \vec{k}_4)$$

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$$
\begin{aligned}
v_c^4(\vec{k}_1, \ldots, \vec{k}_4) &= \exp\left(\frac{\theta}{2i}\,(\vec{k}_1 \times \vec{k}_2 + \vec{k}_3 \times \vec{k}_4)\right)\left((\vec{k}_1 - \vec{k}_2) \cdot \vec{k}_3 + \frac{{m_0}^2}{2}\right) \\
&+ \;(cyclic)
\end{aligned}
\tag{2.7}
$$

$$
\begin{aligned}
S^5[\pi_0] &= \prod_{l=1}^{5}\left(\int \frac{d^2\vec{k}_l}{(2\pi)^2}\,\tilde{\pi}_0^{b_l}(\vec{k}_l)\right)(2\pi)^2\,\delta^2\left(\sum_{l=1}^{5}\vec{k}_l\right)\frac{\alpha_0\,\lambda_0^{\,3}}{i\,300}\,\kappa_{b_1\ldots b_5}\,v_c^5(\vec{k}_1, \ldots, \vec{k}_5) \\
v_c^5(\vec{k}_1, \ldots, \vec{k}_5) &= \exp\left(\frac{\theta}{2i}\,(\vec{k}_1 \times \vec{k}_2 + \vec{k}_1 \times \vec{k}_3 + \vec{k}_2 \times \vec{k}_3 + \vec{k}_4 \times \vec{k}_5)\right)\vec{k}_1 \times (\vec{k}_2 + 3\,\vec{k}_4) \\
&+ \;(cyclic)
\end{aligned}
\tag{2.8}
$$

$$
\begin{aligned}
S^6[\pi_0] &= \prod_{l=1}^{6}\left(\int \frac{d^2\vec{k}_l}{(2\pi)^2}\,\tilde{\pi}_0^{b_l}(\vec{k}_l)\right)(2\pi)^2\,\delta^2\left(\sum_{l=1}^{6}\vec{k}_l\right)\frac{\lambda_0^4}{6!\,3}\,\kappa_{b_1\ldots b_6}\,v_c^6(\vec{k}_1, \ldots, \vec{k}_6) \\
v_c^6(\vec{k}_1, \ldots, \vec{k}_6) &= \exp\left(\frac{\theta}{2\,i}\,(\vec{k}_1 \times \vec{k}_2 + \vec{k}_1 \times \vec{k}_3 + \vec{k}_2 \times \vec{k}_3 + \vec{k}_4 \times \vec{k}_5 + \vec{k}_4 \times \vec{k}_6 + \vec{k}_5 \times \vec{k}_6)\right) \\
&\times\left(\vec{k}_6 \cdot (-3\,\vec{k}_3 + 4\,\vec{k}_4 - \vec{k}_5) + \frac{{m_0}^2}{2}\right) + (cyclic)
\end{aligned}
\tag{2.9}
$$

We have expressed them in terms of vertex functions v_c^n cyclic-invariant in momenta because this form is very useful in the evaluation of Feynmann diagrams. Also the momentum conservation is taken into account to simplify them. [5] We remember finally that loop corrections correspond to powers of λ^2 (see Section 5 for definitions of the renormalized parameters).

3   The one-loop two-points function

In what follows we write a generic OPI correlation function in momentum space as

$$
\Gamma\left(\vec{k}_1, \ldots, \vec{k}_n\right) = (2\pi)^2\,\delta^2\left(\sum_{i=1}^{n}\vec{k}_i\right)\tilde{\Gamma}\left(\vec{k}_1, \ldots, \vec{k}_n\right)
\tag{3.1}
$$

The tree level OPI two-point function [6] is

$$
\tilde{\Gamma}_{a_1 a_2}^{(0)}(p) = \kappa_{a_1 a_2}\,(p^2 + m^2)
\tag{3.2}
$$

At one-loop the contribution to the OPI two-point function consist of two parts that can be visualized as standard graphs; the first one labeled "(a)" contains two three-point vertices and is given by [7]

$$
\tilde{\Gamma}_{a_1 a_2}^{(1a)}(p) = \alpha^2\,\lambda^2\,\left(N\,\kappa_{a_1 a_2}\,f(p; 0) + \kappa_{a_1}\,\kappa_{a_2}\,f(p; \theta)\right)
\tag{3.3}
$$

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[5] Taking into account this constraint the exponential θ-dependent factors result cyclic invariant.

[6] Most precisely the inverse of the connected two-point function; it is OPI for vertex functions, see for example [11].

[7] Indeed there exist a third contribution of this kind, a "blob", that is identically null.

We note the second term in (3.3) is like a "non planar" diagram in the language of [4]. The function $f(p; \theta)$ is exactly computed in Appendix B in various regimes of the parameter θ; its large Λ result is

$$
f(p; 0) = -\frac{p^2}{8\pi}\left(Ei(-\frac{m^2}{\Lambda^2} - \frac{p^2}{4\Lambda^2}) - 1 + \sqrt{1 + \frac{4m^2}{p^2}}\, \ln\frac{4\,m^2}{p^2 + 4\,m^2} + \ln\frac{\sqrt{p^2 + 4\,m^2} + p}{\sqrt{p^2 + 4\,m^2} - p}\right)
$$

(3.4)

and if $\theta \neq 0$ $(\tilde{p}^2 \equiv p\sqrt{p^2 + 4\,m^2})$,

$$
\begin{aligned}
f(p; \theta) &= -\frac{\tilde{p}^2}{8\pi}\left(e^{\frac{\theta\,\tilde{p}^2}{2}}\, Ei\left(\frac{\theta\,\tilde{p}^2}{2}\right) + e^{-\frac{\theta\,\tilde{p}^2}{2}}\, Ei\left(-\frac{\theta\,\tilde{p}^2}{2}\right) + f_m(p; \theta)\right) \\
f_m(p; \theta) &= \frac{4}{\theta\,\tilde{p}^2}\int_0^{\theta\,m\,p} dx\, \frac{x\,(x\,K_0(x))'}{\sqrt{\frac{\theta^2\,\tilde{p}^4}{4} - x^2}} \xrightarrow{m \to 0} 0
\end{aligned}
$$

(3.5)

where K_0 is a Bessel function.

The second contribution labeled "(b)" comes from a diagram containing a quartic vertex with a self-contraction and is given by

$$
\begin{aligned}
\tilde{\Gamma}^{(1b)}_{a_1 a_2}(p) &= -\frac{\lambda^2}{6}\left(N\,\kappa_{a_1 a_2}\left(g_1(p; 0) + (p^2 + 2m^2)\,g_2(p; 0)\right)\right. \\
&\left.\quad + \kappa_{a_1}\,\kappa_{a_2}\left(g_1(p; \theta) + (p^2 - m^2)\,g_2(p; \theta)\right)\right) \\
g_i(p; \theta) &= \int \frac{d^2\vec{k}}{4\pi^2}\, \tilde{G}(k)\, e^{i\,\theta\,\vec{p}\times\vec{k}} \times \begin{cases} k^2 & , \quad i = 1 \\ 1 & , \quad i = 2 \end{cases}
\end{aligned}
$$

(3.6)

The functions g_i are easily computed as we made with $f(p; \theta)$ in Appendix B; for large Λ we have

$$
\begin{aligned}
g_1(p; \theta) &= \begin{cases} \frac{1}{4\pi}\left(2\Lambda^2 + m^2\,Ei(-\frac{m^2}{2\Lambda^2})\right) & , \quad \theta\,p = 0 \\ \frac{1}{\theta^2}\,\delta^2(\vec{p}) - \frac{m^2}{2\pi}\,K_0(\theta\,m\,p) & , \quad \theta\,p \neq 0 \end{cases} \\
g_2(p; \theta) &= \begin{cases} -\frac{1}{4\pi}\,Ei(-\frac{m^2}{2\Lambda^2}) & , \quad \theta\,p = 0 \\ \frac{1}{2\pi}\,K_0(\theta\,m\,p) & , \quad \theta\,p \neq 0 \end{cases}
\end{aligned}
$$

(3.7)

Some remarks are in order. In first place we note that as remarked in Appendix B, $f(0; \theta) = 0$ and the limits $\Lambda \to \infty$ and $p = 0$ commute; this is due to the factorization of the external momentum p coming from the vertex. However it is not the case for the tadpole diagram where we see from (3.7) that the presence of the scale θ induces, at fixed Λ, standard logarithmic singularities in $p = 0$ and a δ-type singularity; similar term was recently noted in [5] in the context of a 2+1 dimensional non-relativistic non-commutative field theory; an analogous term is shown to be present in the relativistic theory in a certain limit. It is also remarked there that it is not possible to apply the usual normal ordering prescription of commutative field theories to set the tadpole to zero because of its dependence on the external momentum coming from the non-planar diagrams. However they will be irrelevant to the computation of the renormalization group functions as will be shown in detail in Section 5. In second place, we should hope in the commutative case

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to have only contributions proportional to the traceless part corresponding to $SU(N)$ because the $U(1)$ trace part is just a free field, i.e. the correlation function should be proportional to the tensor

$$\kappa_{a_1 a_2}^{(tr)} = \kappa_{a_1 a_2} + \frac{\kappa_{a_1}\,\kappa_{a_2}}{N} \quad , \quad \kappa^{a_1 a_2}\,\kappa_{a_1 a_2}^{(tr)} = 0 \tag{3.8}$$

This is evident in (3.3), not so in (3.6), however it becomes true in the massless limit. The explanation of this fact lies in the presence of the mass term (2.3) that introduces interactions for the trace part not present in the classical theory.

4   The one-loop three-points function

From (2.9) we read that the OPI three-point function at tree level is given by

$$\tilde{\Gamma}_{a_1 a_2 a_3}^{(0)}(\vec{p}) = -\frac{i\,\alpha\,\lambda}{3}\,\kappa_{a_1 a_2 a_3}\,\vec{p}_1 \times \vec{p}_2\,e^{-i\frac{\theta}{2}\vec{p}_1 \times \vec{p}_2} + (perm.) \tag{4.1}$$

where "(perm.)" implies all permutation terms of external legs. Standard series expansion tells us that at one loop there exist three contributions. The first labeled "(a)" is a tadpole with a five-points vertex insertion,

$$\tilde{\Gamma}_{a_1 a_2 a_3}^{(1a)}(\vec{p}) = \frac{i\,\alpha\,\lambda^3}{12}\,\vec{p}_1 \times \vec{p}_2\,e^{-i\frac{\theta}{2}\vec{p}_1 \times \vec{p}_2}\,\left(N\,\kappa_{a_1 a_2 a_3}\,g_2(p_3, 0)\; + \kappa_{a_1 a_2}\kappa_{a_3}\,g_2(p_3; \theta)\right) + (perm.) \tag{4.2}$$

where $g_2(p; \theta)$ is given in (3.7). The second contribution labeled "(b)" contains a three-points vertex and a four-points vertex and is given by

$$
\begin{aligned}
\tilde{\Gamma}_{a_1 a_2 a_3}^{(1b)}(\vec{p}) &= \frac{i\,\alpha\,\lambda^3}{12}\,e^{-i\frac{\theta}{2}\vec{p}_1 \times \vec{p}_2}\int \frac{d^2\vec{k}}{4\,\pi^2}\,\frac{e^{-\frac{1}{2\Lambda^2}(\vec{k}^2 + (\vec{k} - \vec{p}_1)^2)}}{(\vec{k}^2 + m^2)\,((\vec{k} - \vec{p}_1)^2 + m^2)}\,\vec{p}_1 \times \vec{k} \\
&\times \left(N\,\kappa_{a_1 a_2 a_3}\,P_1(\vec{k}; \vec{p}) + \kappa_{a_1}\,\kappa_{a_2 a_3}\,e^{-i\theta\,\vec{p}_1 \times \vec{k}}\,P_1(\vec{k}; \vec{p})\right. \\
&+ \left.\kappa_{a_2}\,\kappa_{a_3 a_1}\,e^{-i\theta\,\vec{p}_2 \times \vec{k}}\,P_2(\vec{k}; \vec{p}) + \kappa_{a_3}\,\kappa_{a_1 a_2}\,e^{-i\theta\,\vec{p}_3 \times \vec{k}}\,P_2(\vec{k}; \vec{p})\right) \\
&+ (perm.)
\end{aligned}
\tag{4.3}
$$

where

$$
\begin{aligned}
P_1(\vec{k}; \vec{p}) &= \vec{k}^2 + 2\,\vec{k} \cdot (\vec{p}_1 + 3\,\vec{p}_3) + \vec{p}_3^{\,2} + 2\,\vec{p}_1 \cdot \vec{p}_2 + 2\,m^2 \\
P_2(\vec{k}; \vec{p}) &= \vec{k}^2 - \vec{k} \cdot \vec{p}_1 - \frac{\vec{p}_1^{\,2}}{2} - \vec{p}_2 \cdot \vec{p}_3 - m^2
\end{aligned}
\tag{4.4}
$$

Finally the third contribution labeled "(c)" contains three three-point vertices,

$$
\begin{aligned}
\tilde{\Gamma}_{a_1 a_2 a_3}^{(1c)}(\vec{p}) &= -\frac{i\,(\alpha\,\lambda)^3}{3}\,e^{-i\frac{\theta}{2}\vec{p}_1 \times \vec{p}_2}\,\left(N\,\kappa_{a_1 a_2 a_3}\,F(\vec{p}; 0) + 3\,\kappa_{a_1}\,\kappa_{a_2 a_3}\,F(\vec{p}; \theta)\right) \\
&+ (perm.)
\end{aligned}
$$

$$F(\vec{p};\theta) = \int \frac{d^2\vec{k}}{4\pi^2} \frac{e^{-\frac{1}{2\Lambda^2}\left(\vec{k}^2+(\vec{k}+\vec{p}_1)^2+(\vec{k}-\vec{p}_2)^2\right)+i\,\theta\,\vec{p}_1\times\vec{k}}}{(\vec{k}^2+m^2)\left((\vec{k}+\vec{p}_1)^2+m^2\right)\left((\vec{k}-\vec{p}_2)^2+m^2\right)}$$
$$\times \quad \vec{p}_1\times\vec{k}\;\; \vec{p}_2\times\vec{k}\;\; \vec{p}_3\times(\vec{k}+\vec{p}_1) \tag{4.5}$$

The integrals are straightforwardly evaluated as in Appendix B; in order not to be reiterative we present the large Λ result of (b) useful in the next section

$$\tilde{\Gamma}^{(1b)}_{a_1a_2a_3}(\vec{p}) = -\frac{3\,N\,\lambda^2}{16\,\pi}\,\ln\frac{m^2}{\Lambda^2}\,\tilde{\Gamma}^{(0)}_{a_1a_2a_3}(\vec{p}) \tag{4.6}$$

In what contribution (c) concerns we just say that is IR and UV *finite*, the reason behind being the derivative vertices present and the factorization of powers of external momenta respectively.

5   Renormalization

As a by-product of the computations made we will obtain here the functions of the renormalization group working in the Callan-Symanzik context [12]. In order to get rid of the divergences we introduce renormalized fields and constants that will define the counter-terms in the way

$$\begin{aligned}
\pi_0(\vec{x}) &\equiv Z^{\frac{1}{2}}\,\pi(\vec{x})\\
m_0{}^2 &\equiv Z^{-1}\,Z_m\,m^2\\
\lambda_0 &\equiv Z^{-1}\,Z_\lambda{}^{\frac{1}{2}}\,\lambda\\
\alpha_0 &\equiv Z^{-\frac{1}{2}}\,Z_\lambda{}^{-\frac{1}{2}}\,Z_\alpha\,\alpha
\end{aligned} \tag{5.1}$$

Let us assume k is not renormalized; in the next section we will prove it. Then the relation

$$Z_\lambda = Z\,Z_\alpha{}^{\frac{2}{3}} \tag{5.2}$$

should hold. So hopefully three renormalization constants Z, Z_λ and Z_m will be enough to make finite the theory. All renormalized quantities will depend on a scale of renormalization μ at which they will be defined. Let us introduce therefore the renormalization group functions

$$\begin{aligned}
\beta &\equiv \mu\frac{d\lambda}{d\mu}\\
\gamma_m &\equiv \mu\frac{d\ln m^2}{d\mu}\\
\gamma_\alpha &\equiv \mu\frac{d\alpha}{d\mu}
\end{aligned} \tag{5.3}$$

By imposing as usual the independence of μ of the bare parameters we get from (5.1), (5.3) the algebraic relation

$$\gamma_\alpha = \frac{2\,\alpha}{\lambda}\,\beta \tag{5.4}$$

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together with the system of equations

$$
\left(\lambda \frac{d \ln(Z_m/Z)}{d\lambda} + 2\,\alpha \frac{d \ln(Z_m/Z)}{d\alpha} \right) \frac{\beta}{\lambda} + \left(1 + m^2 \frac{d \ln(Z_m/Z)}{dm^2} \right) \gamma_m = \mu \frac{d \ln(Z/Z_m)}{d\mu}
$$
$$
\left(1 + \lambda \frac{d \ln(Z_\lambda^{\frac{1}{2}}/Z)}{d\lambda} + 2\,\alpha \frac{d \ln(Z_\lambda^{\frac{1}{2}}/Z)}{d\alpha} \right) \frac{\beta}{\lambda} + m^2 \frac{d \ln(Z_\lambda^{\frac{1}{2}}/Z)}{dm^2} \gamma_m = \mu \frac{d \ln(Z/Z_\lambda^{\frac{1}{2}})}{d\mu}
$$
$$(5.5)$$

This is a inhomogeneous linear system with straightforward solution; at one loop

$$
\beta|_{1l} = \lambda\,\mu \frac{d \ln(Z/Z_\lambda^{\frac{1}{2}})}{d\mu}
$$
$$
\gamma_m|_{1l} = \mu \frac{d \ln(Z/Z_m)}{d\mu}
\tag{5.6}
$$

Now from (2.5), (5.1) we obtain counter-term contributions to the two and three-point functions of the form

$$
\tilde{\Gamma}_{a_1 a_2}^{(1ct)}(p) = \left((Z-1)\,p^2 + (Z_m - 1)\,m^2 \right)\,\kappa_{a_1 a_2}
$$
$$
\tilde{\Gamma}_{a_1 a_2 a_3}^{(1ct)}(\vec{p}) = (Z_\alpha - 1)\,\tilde{\Gamma}_{a_1 a_2 a_3}^{(0)}(\vec{p})
\tag{5.7}
$$

To renormalize the theory, from (3.3)-(3.7) we fix at arbitrary μ

$$
Z|_{1l} = 1 + \frac{N\,\lambda^2}{24\,\pi} \left(1 - 3\,\alpha^2 \right)\,\ln \frac{\Lambda^2}{\mu^2}
$$
$$
Z_m|_{1l} = 1 + \frac{N\,\lambda^2}{24\,\pi} \left(2\,\frac{\Lambda^2}{m^2} + \ln \frac{\Lambda^2}{\mu^2} \right)
\tag{5.8}
$$

Similarly for the three-point function and taking into account the finiteness of contribution "(c)" we fix from (4.2), (4.6)

$$
Z_\alpha|_{1l} = 1 + \frac{N\lambda^2}{16\,\pi}\,\ln \frac{\Lambda^2}{\mu^2} - \frac{3\,N\lambda^2}{16\,\pi}\,\ln \frac{\Lambda^2}{\mu^2}
\tag{5.9}
$$

and admitting (5.2) holds we read

$$
Z_\lambda|_{1l} = 1 + \frac{1}{2}\,(Z-1) + \frac{1}{3}\,(Z_\alpha - 1) = 1 - \frac{N\,\lambda^2}{24\,\pi}\,(1 + 3\,\alpha^2)\,\ln \frac{\Lambda^2}{\mu^2}
\tag{5.10}
$$

From (5.4), (5.6) we finally get

$$
\beta|_{1l} = -\frac{N\,\lambda^3}{8\,\pi}\,(1 - \alpha^2)
$$
$$
\gamma_m|_{1l} = \frac{N\,\lambda^2\,\alpha^2}{4\,\pi}
$$

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$$\gamma_\alpha|_{1l} = -\frac{N\,\lambda^2\,\alpha}{4\,\pi}\,(1-\alpha^2) \tag{5.11}$$

They are essentially the same as in the commutative case but applied to the whole fields; as we have pointed before in the commutative case they refer just to the $SU(N)$ part. It is worth to note that the contributions to (5.11) coming from the tadpole diagrams from the two-point and three-point correlators exactly cancel leaving the contributions from the graphs (a) and (b) respectively. Furthermore if we introduce the renormalization constant Z_g by [10]

$$\frac{m_0{}^2}{\lambda_0{}^2} \equiv Z_g\,\frac{m^2}{\lambda^2} \quad\leftrightarrow\quad Z_g = Z_\lambda{}^{-1}\,Z\,Z_m \overset{1l}{=} 1 + \frac{N\,\lambda^2}{24\,\pi}\,(2\,\frac{\Lambda^2}{m^2} + 3\,\ln\frac{\Lambda^2}{\mu^2}) \tag{5.12}$$

then the anomalous dimension for the field g is given by

$$\gamma_g \equiv -\mu\frac{d\ln Z_g}{d\mu} \overset{1l}{=} \frac{N\,\lambda^2}{4\,\pi} \overset{\alpha=1}{\longrightarrow} \frac{N}{k} \tag{5.13}$$

as we could hope from current algebra representation theory for a field transforming in the $U(N)$ fundamental representation in the critical model [13].

6   The one-loop four-points function

In this section we will compute the coupling constant renormalization from the four-points function verifying that (5.2) indeed holds.

The tree level vertex is

$$\begin{aligned}
\tilde{\Gamma}^{(0)}_{a_1\ldots a_4}(\vec{p}) &= \frac{\lambda^2}{48}\,\kappa_{a_1\ldots a_4}\,e^{E_4(\vec{p})}\,(\Gamma(\vec{p}) + 2\,m^2) + (perm.) \\
E_4(\vec{p}) &= \exp\left(\frac{\theta}{2i}\,(\vec{p}_1 \times \vec{p}_2 + \vec{p}_3 \times \vec{p}_4)\right) \\
\Gamma(\vec{p}) &= 2\,u^2 - s^2 - t^2
\end{aligned} \tag{6.1}$$

where we have introduced $\vec{s} \equiv \vec{p}_1 + \vec{p}_2$, $\vec{u} \equiv \vec{p}_1 + \vec{p}_3$, $\vec{t} \equiv \vec{p}_1 + \vec{p}_4$, $s^2 + t^2 + u^2 = \sum_{i=1}^4 p_i{}^2$.

At one-loop there are six contributions to this correlator that we write below with their divergent parts.

The contribution "(a)" is a tadpole diagram with a six-points vertex,

$$\begin{aligned}
\tilde{\Gamma}^{(1a)}_{a_1\ldots a_4}(\vec{p}) &= \frac{\lambda^4}{5!\,3}\left(-N\,\kappa_{a_1\ldots a_4}\,\Gamma^{(1a)}_1(\vec{p}) + \kappa_{a_1}\,\kappa_{a_2 a_3 a_4}\,\Gamma^{(1a)}_2(\vec{p}) + \kappa_{a_1 a_2}\,\kappa_{a_3 a_4}\,\Gamma^{(1a)}_3(\vec{p})\right) \\
&+ (perm.) \\
\Gamma^{(1a)}_i(\vec{p}) &= \int \frac{d^2\vec{k}}{4\,\pi^2}\,\tilde{G}(k^2)\left\{\begin{array}{ll} v_c^6(-\vec{k},\vec{k},\vec{p}_1,\ldots,\vec{p}_4) & ,\ i=1 \\ v_c^6(-\vec{k},\vec{p}_1,\vec{k},\vec{p}_2,\vec{p}_3,\vec{p}_4) & ,\ i=2 \\ \frac{1}{2}\,v_c^6(-\vec{k},\vec{p}_1,\vec{p}_2,\vec{k},\vec{p}_3,\vec{p}_4) & ,\ i=3 \end{array}\right.
\end{aligned} \tag{6.2}$$

Its divergent part is given by

$$
\begin{aligned}
\tilde{\Gamma}^{(1a)}_{a_1\ldots a_4}(\vec{p})|_{div.} &= \frac{N\,\lambda^4}{48\cdot 12\pi}\,\kappa_{a_1\ldots a_4}\,e^{E_4(\vec{p})}\left(-\frac{4}{5}\,\Lambda^2 + (\Gamma(\vec{p})+2\,m^2)\,\ln\frac{m^2}{\Lambda^2}\right.\\
&\quad + \left.(-\frac{6}{5}\,m^2 + \frac{1}{5}(s^2+t^2) - \frac{3}{10}\,u^2)\ln\frac{m^2}{\Lambda^2}\right) + (perm.)
\end{aligned}\tag{6.3}
$$

The contribution "(b)" is a diagram with two four-points vertices,

$$
\begin{aligned}
\tilde{\Gamma}^{(1b)}_{a_1\ldots a_4}(\vec{p}) &= -\frac{\lambda^4}{4!\,12}\left(-N\,\kappa_{a_1\ldots a_4}\,\Gamma^{(1b)}_1(\vec{p}) + \kappa_{a_1}\,\kappa_{a_2 a_3 a_4}\,\Gamma^{(1b)}_2(\vec{p}) + \kappa_{a_1 a_2}\,\kappa_{a_3 a_4}\,\Gamma^{(1b)}_3(\vec{p})\right.\\
&\quad + \left.\kappa_{a_1 a_4}\,\kappa_{a_2 a_3}\,\Gamma^{(1b)}_4(\vec{p})\right) + (perm.)
\end{aligned}
$$

$$
\Gamma^{(1b)}_i(\vec{p}) = \int\frac{d^2\vec{k}}{4\,\pi^2}\,\tilde{G}(k^2)\,\tilde{G}((\vec{k}-\vec{s})^2)
\begin{cases}
v^4_c(\vec{p}_1,\vec{p}_2,\vec{k},-\vec{k}-\vec{s})\,v^4_c(\vec{p}_3,\vec{p}_4,-\vec{k}+\vec{s},\vec{k})\\
2\,v^4_c(\vec{p}_1,\vec{p}_2,-\vec{k},\vec{k}-\vec{s})\,v^4_c(\vec{p}_3,\vec{k},\vec{p}_4,-\vec{k}+\vec{s})\\
v^4_c(\vec{p}_1,\vec{p}_2,-\vec{k},\vec{k}-\vec{s})\,v^4_c(\vec{p}_3,\vec{p}_4,\vec{k},-\vec{k}+\vec{s})\\
\frac{1}{2}\,v^4_c(\vec{p}_1,-\vec{k},\vec{p}_2,\vec{k}-\vec{s})\,v^4_c(\vec{p}_3,\vec{k},\vec{p}_4,-\vec{k}+\vec{s})
\end{cases}\tag{6.4}
$$

for $i = 1, 2, 3, 4$ respectively. Its divergent part is given by

$$
\begin{aligned}
\tilde{\Gamma}^{(1b)}_{a_1\ldots a_4}(\vec{p})|_{div.} &= \frac{N\,\lambda^4}{48\cdot 12\pi}\,\kappa_{a_1\ldots a_4}\,e^{E_4(\vec{p})}\left(\frac{1}{2}\,\Lambda^2 - \frac{3}{2}\,(\Gamma(\vec{p})+2\,m^2)\,\ln\frac{m^2}{\Lambda^2}\right.\\
&\quad + \left.(2\,m^2 + \frac{1}{2}\,u^2)\ln\frac{m^2}{\Lambda^2}\right) + (perm.)
\end{aligned}\tag{6.5}
$$

The contribution "(c)" is a diagram with a three-points vertex and a four-points vertex,

$$
\begin{aligned}
\tilde{\Gamma}^{(1c)}_{a_1\ldots a_4}(\vec{p}) &= \frac{\alpha^2\,\lambda^4}{180}\left(-N\,\kappa_{a_1\ldots a_4}\,\Gamma^{(1c)}_1(\vec{p}) + \kappa_{a_1 a_2}\,\kappa_{a_3 a_4}\,\Gamma^{(1c)}_2(\vec{p}) + \kappa_{a_4}\,\kappa_{a_1 a_2 a_3}\,\Gamma^{(1c)}_3(\vec{p})\right.\\
&\quad + \left.\kappa_{a_3}\,\kappa_{a_1 a_2 a_4}\,\Gamma^{(1c)}_4(\vec{p})\right) + (perm.)
\end{aligned}
$$

$$
\Gamma^{(1c)}_i(\vec{p}) = \int\frac{d^2\vec{k}}{4\,\pi^2}\,\tilde{G}(k^2)\,\tilde{G}((\vec{k}+\vec{p}_4)^2)
\begin{cases}
v^3_c(\vec{k},\vec{p}_4,-\vec{k}-\vec{p}_4)\,v^5_c(\vec{p}_1,\vec{p}_2,\vec{p}_3,-\vec{k},\vec{k}+\vec{p}_4)\\
v^3_c(\vec{p}_4,\vec{k},-\vec{k}-\vec{p}_4)\,v^5_c(\vec{p}_1,\vec{p}_2,-\vec{k},\vec{p}_3,\vec{k}+\vec{p}_4)\\
v^3_c(\vec{p}_4,\vec{k},-\vec{k}-\vec{p}_4)\,v^5_c(\vec{p}_1,\vec{p}_2,\vec{p}_3,-\vec{k},\vec{k}+\vec{p}_4)\\
v^3_c(\vec{k},\vec{p}_4,-\vec{k}-\vec{p}_4)\,v^5_c(\vec{p}_1,\vec{p}_2,-\vec{k},\vec{p}_3,\vec{k}+\vec{p}_4)
\end{cases}\tag{6.6}
$$

for $i = 1, 2, 3, 4$ respectively. Its divergent part is given by

$$
\tilde{\Gamma}^{(1c)}_{a_1\ldots a_4}(\vec{p})|_{div.} = \frac{N\,\alpha^2\lambda^4}{48\cdot 4\pi}\,\kappa_{a_1\ldots a_4}\,e^{E_4(\vec{p})}\,(-\Gamma(\vec{p}) + \vec{p}_1\cdot\vec{p}_3 + \vec{p}_2\cdot\vec{p}_4)\,\ln\frac{m^2}{\Lambda^2} + (perm.)\tag{6.7}
$$

The contribution "(d)" is a diagram with two three-points vertices and one four-points vertex,

$$
\tilde{\Gamma}^{(1d)}_{a_1\ldots a_4}(\vec{p}) = -\frac{\alpha^2\,\lambda^4}{108}\left(-N\,\kappa_{a_1\ldots a_4}\,\Gamma^{(1d)}_1(\vec{p}) + \kappa_{a_1 a_2}\,\kappa_{a_3 a_4}\,\Gamma^{(1d)}_2(\vec{p}) + \kappa_{a_3}\,\kappa_{a_1 a_2 a_4}\,\Gamma^{(1d)}_3(\vec{p})\right.
$$

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$$
\begin{aligned}
+ \quad & \kappa_{a_4}\,\kappa_{a_1 a_2 a_3}\,\Gamma_4^{(1d)} + \kappa_{a_2}\,\kappa_{a_1 a_3 a_4}\,\Gamma_5^{(1d)} + \kappa_{a_1}\,\kappa_{a_2 a_4 a_3}\,\Gamma_6^{(1d)} + \kappa_{a_1 a_4}\,\kappa_{a_2 a_3}\,\Gamma_7^{(1d)} \\
+ \quad & \kappa_{a_1 a_3}\,\kappa_{a_2 a_4}\,\Gamma_8^{(1d)}\Big) + (perm.)
\end{aligned}
$$

$$
\Gamma_i^{(1d)}(\vec{p}) \;=\; \int \frac{d^2\vec{k}}{4\,\pi^2}\,\tilde{G}(k^2)\,\tilde{G}((\vec{k}-\vec{s})^2)\,\tilde{G}((\vec{k}+\vec{p}_3)^2)\,v_c^4(\vec{p}_1,\vec{p}_2,-\vec{k},\vec{k}-\vec{s})
$$

$$
\begin{cases}
v_c^3(\vec{k},\vec{p}_3,-\vec{k}-\vec{p}_3)\,v_c^3(\vec{k}+\vec{p}_3,\vec{p}_4,-\vec{k}+\vec{s}) & ,\quad i=1 \\
v_c^3(\vec{p}_3,\vec{k},-\vec{k}-\vec{p}_3)\,v_c^3(\vec{p}_4,\vec{k}+\vec{p}_3,-\vec{k}+\vec{s}) & ,\quad i=2 \\
v_c^3(\vec{p}_3,\vec{k},-\vec{k}-\vec{p}_3)\,v_c^3(\vec{k}+\vec{p}_3,\vec{p}_4,-\vec{k}+\vec{s}) & ,\quad i=3 \\
v_c^3(\vec{k},\vec{p}_3,-\vec{k}-\vec{p}_3)\,v_c^3(\vec{p}_4,\vec{k}+\vec{p}_3,-\vec{k}+\vec{s}) & ,\quad i=4
\end{cases}
$$

$$(6.8)$$

while for $i = 5, 6, 7, 8$ the expression is similar with $v_c^4(\vec{p}_1,\vec{p}_2,-\vec{k},\vec{k}-\vec{s})$ replaced by $\frac{1}{2} v_c^4(\vec{p}_1,-\vec{k},\vec{p}_2,\vec{k}-\vec{s})$. Its divergent part is given by

$$
\tilde{\Gamma}_{a_1...a_4}^{(1d)}(\vec{p})|_{div.} = \frac{N\,\alpha^2\lambda^4}{48\cdot 4\pi}\,\kappa_{a_1...a_4}\,e^{E_4(\vec{p})}\,\left(\frac{1}{2}\Gamma(\vec{p}) - \vec{p}_1\cdot\vec{p}_3 - \vec{p}_2\cdot\vec{p}_4\right)\,\ln\frac{m^2}{\Lambda^2} + (perm.) \quad (6.9)
$$

The contribution "(e)" is a diagram with four three-points vertices,

$$
\begin{aligned}
\tilde{\Gamma}_{a_1...a_4}^{(1e)}(\vec{p}) \;=\; & -\frac{\alpha^4\,\lambda^4}{81}\Big(\kappa_{a_1}\,\kappa_{a_2 a_4 a_3}\,\Gamma_1^{(1e)}(\vec{p}) + \kappa_{a_1 a_4}\,\kappa_{a_2 a_3}\,\Gamma_2^{(1e)}(\vec{p}) + \kappa_{a_1 a_3}\,\kappa_{a_2 a_4}\,\Gamma_3^{(1e)}(\vec{p}) \\
& -\; N\,\kappa_{a_1 a_4 a_3 a_2}\,\Gamma_4^{(1e)}(\vec{p}) +\Big) + (perm.)
\end{aligned}
$$

$$
\Gamma_i^{(1e)}(\vec{p}) \;=\; \int \frac{d^2\vec{k}}{4\,\pi^2}\,\tilde{G}(k^2)\,\tilde{G}((\vec{k}-\vec{p}_1)^2)\,\tilde{G}((\vec{k}+\vec{p}_2)^2)\,\tilde{G}((\vec{k}-\vec{u})^2)\,v_c^3(\vec{p}_2,\vec{k},-\vec{k}-\vec{p}_2)
$$

$$
\begin{cases}
v_c^3(\vec{p}_1-\vec{k},\vec{p}_4,\vec{k}-\vec{t})\,v_c^3(\vec{p}_3,\vec{k}+\vec{p}_2,-\vec{k}+\vec{t})\,v_c^3(-\vec{p}_1,\vec{k},-\vec{k}+\vec{p}_1) \\
\frac{1}{2}\,v_c^3(\vec{p}_4,\vec{p}_1-\vec{k},\vec{k}-\vec{t})\,v_c^3(\vec{p}_3,\vec{k}+\vec{p}_2,-\vec{k}+\vec{t})\,v_c^3(-\vec{p}_1,\vec{k},-\vec{k}+\vec{p}_1) \\
\frac{1}{4}\,v_c^3(\vec{p}_1-\vec{k},\vec{p}_4,\vec{k}-\vec{t})\,v_c^3(\vec{k}+\vec{p}_2,\vec{p}_3,-\vec{k}+\vec{t})\,v_c^3(-\vec{p}_1,\vec{k},-\vec{k}+\vec{p}_1) \\
\frac{1}{4}\,v_c^3(\vec{p}_1-\vec{k},\vec{p}_4,\vec{k}-\vec{t})\,v_c^3(\vec{p}_3,\vec{k}+\vec{p}_2,-\vec{k}+\vec{t})\,v_c^3(\vec{k},-\vec{p}_1,-\vec{k}+\vec{p}_1)
\end{cases}
$$

$$(6.10)$$

for $i = 1, 2, 3, 4$ respectively. This contribution, as "(c)" in the three-points function, is UV and IR finite for general external momenta.

Finally from the definitions (5.1) we get the counter-term contribution

$$
\tilde{\Gamma}_{a_1...a_4}^{(1ct)}(\vec{p}) = (Z_\lambda - 1)\,\tilde{\Gamma}_{a_1...a_4}^{(0)}(\vec{p}) + \frac{Z_\lambda}{Z}\,(Z_m - Z)\,\frac{m^2\,\lambda^2}{4!}\,\kappa_{a_1...a_4}\,e^{E_4(\vec{p})} + (perm.) \quad (6.11)
$$

By summing up all the contributions we arrive to the result

$$
\begin{aligned}
\tilde{\Gamma}_{a_1...a_4}^{(1l)}(\vec{p})|_{div.} \;=\; & \frac{N\,\lambda^4}{48\cdot 12\,\pi}\,\kappa_{a_1...a_4}\,e^{E_4(\vec{p})}\,\left(\frac{17}{10}\Lambda^2 + \frac{1}{5}\sum_{i=1}^{4}(p_i{}^2 + m^2)\right) + (perm.) \\
& +\; \left(Z_\lambda|_{1l} - 1 - \frac{N\,\lambda^2}{24\,\pi}\,(1 + 3\,\alpha^2)\,\ln\frac{m^2}{\Lambda^2}\right)\,\tilde{\Gamma}_{a_1...a_4}^{(0)}(\vec{p}) \quad (6.12)
\end{aligned}
$$

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The second line line fixes the one-loop value of Z_λ exactly to that given in (5.2) verifying in this way the non-renormalization of the level k, while the second term in the first line needs a non left-right invariant counter-term of the form

$$\delta S[\pi] = \frac{N \lambda^4}{48 \cdot 15 \pi} \ln \frac{\Lambda^2}{\mu^2} \int d^2\vec{x}\, Tr\left(\pi^3\,(-\Box + m^2)\pi\right) \tag{6.13}$$

which however is identically zero on-shell!

7 Conclusions

We have computed in a Schwinger regularization scheme (not usual to my knowledge in the literature on the subject) correlation functions in a non commutative version of the two dimensional WZW model, where other than the mass parameter another dimensional one, the θ parameter, is present. The results display some features common to other field theories with non-derivative vertices, in particular the different behavior of them depending on the range of the parameters as made explicit in Appendix B and (3.7), which is at the origin of the so-called UV/IR mixing. As a by product we computed the renormalization group functions in a setting less involved than the dimensional regularization context; to this respect we would like to spend some words about the background field method. In order to carry out computations in this context from (2.1) and taking the standard splitting other than the vertices seen new ones appear from (1.3)

$$\frac{1}{\lambda^2} \int d^2\vec{x}\, Tr\left(P_-^{ij}\, L_i(g_0)\, R_j(e^{\lambda \pi})\right) \tag{7.1}$$

The relevant term at one loop coming from (7.1) is

$$V_2 = \frac{1}{2} \int d^2\vec{x}\, Tr\left(P_-^{ij}\, L_i(g_0)\, [\partial_j \pi, \pi]_*\right) \tag{7.2}$$

Then the quadratic contribution to the effective action is given by

$$\begin{aligned}
\Gamma_2^{(1)}[g_0] &= -\frac{1}{2} < V_2\, V_2 >_0 \\
&= \frac{1}{8} P_-^{ij}\, P_-^{kl} \int d^2\vec{x} \int d^2\vec{y}\, L_i^{a_1}(g_0)|_{\vec{x}}\, L_k^{a_2}(g_0)|_{\vec{x}-\vec{y}}\, (-N\, \kappa_{a_1 a_2}\, f_{jl}(y;0) \\
&\quad -\, \kappa_{a_1}\, \kappa_{a_2}\, f_{jl}(y;\theta)) \\
f_{jl}(y;\theta) &= \int \frac{d^2\vec{k}_1}{4\pi^2} \int \frac{d^2\vec{k}_2}{4\pi^2}\, \tilde{G}(k_1)\, \tilde{G}(k_2)\, (\vec{k}_1 - \vec{k}_2)_j\, (\vec{k}_1 - \vec{k}_2)_l\, e^{i(\vec{k}_1+\vec{k}_2)\cdot\vec{y} - i\theta\vec{k}_1\times\vec{k}_2} \tag{7.3}
\end{aligned}$$

This is essentially the computation carried out recently in reference [9] (and time ago in [6] in the commutative case); we do not reproduce the details here, just to say that as verification if we carry out (7.3) and renormalize we get exactly the beta function giving in (5.11). [8] The fact that we obtain the same non-trivial fixed point at the critical point

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[8] We should say that we do not agree in the denominator with the result of reference [9] that is 32π; we get 8π instead.

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$\alpha = 1$ as well as the right conformal dimension in the fundamental representation for the field g seems to indicate that both theories in the critical point could be equivalent. This fact is supported also for the non renormalization of k here verified explicitly and by recent computations of fermionic determinants [14] and a kind of Seiberg-Witten map recently proposed [15] that would prove the equivalence. If this is so then we would certainly have examples of *unitary* field theories NC in time directions, a very different behaviour of non derivative scalar theories recently studied in [16].

In what the four-points correlator computed in Section 6 concerns we would like to comment a couple of things. The first one is the presence of quadratic divergences (c.f equation (6.12)), a natural fact in our regularization context. They should be killed by terms coming from a left-right invariant measure [17], [18]; this is a very subtle subject in the commutative case and it is more in the non-commutative context; we just say that we have carried out all the computations made above in the dimensional regularization scheme of reference [10] where the measure problems are not present [19] because of the identity

$$\int d^d \vec{k} = 0 \quad , \quad d \equiv 2 + 2\epsilon \tag{7.4}$$

and we have obtained exactly the same results presented here with the replacement of $\ln \Lambda^2$ with $1/\epsilon$ and the absence of quadratic divergences.

The second one is the need of (6.13) to renormalize off-shell the theory. While is worth to note that (no) tadpole contributions to Z_λ in (6.12) correspond to (no) tadpole contributions in (5.2), both of them conspire in a non trivial way to give this term. We remark that it must be present in the commutative case also, and what is more, at the level of the non-linear sigma-model obtained by putting $\alpha \equiv 0$, because it comes from the only two contributions (the diagrams "(a)" and "(b)") to the four-point correlator with no odd vertices. Its presence has the same origin as the presence of non-covariant terms in the effective action of general two-dimensional sigma-models noted in references [20], [21], when the expansion is made using a non-covariant field; in fact it is easy to show that in the present context we can absorbe it by the field redefinition

$$\pi(\vec{x}) = \xi(\vec{x}) + \frac{N \lambda^4}{48 \cdot 30 \, \pi} \, \xi(\vec{x})^3 + o(\xi^5) \tag{7.5}$$

Acknowledgements

A   Conventions

We resume here the conventions adopted and some useful formulae.

The Moyal product is defined by

$$f * g\,(\vec{x}) \equiv \exp\left(\frac{i}{2}\,\theta^{\mu\nu}\,\frac{\partial}{\partial y^\mu}\,\frac{\partial}{\partial z^\mu}\right)\,f(\vec{y})\,g(\vec{z})\,|_{\vec{y}=\vec{z}=\vec{x}} \tag{A.1}$$

The space of definition is $\Re^d$ because only on this manifold definition (A.1) has a covariant meaning in cartesian coordinates. [9] The functions are taken to be of integrable square; then in momentum space we have

$$f_1 * \ldots * f_m\,(\vec{x}) = \int \frac{d^d\vec{k}_1}{(2\pi)^d}\cdots\frac{d^d\vec{k}_m}{(2\pi)^d}\,\tilde{f}_1(\vec{k}_1)\ldots\tilde{f}_m(\vec{k}_m)\,e^{i\,\vec{x}\cdot\sum_{l=1}^m \vec{k}_l - \frac{i}{2}\sum_{l<s=1}^m \theta^{\mu\nu}\,k_{l\mu}\,k_{s\nu}} \tag{A.2}$$

It is evident that the reality of $\theta^{\mu\nu}$ is a necessary condition for the existence of the Moyal product. On the other hand the anti-symmetry of it allows to write

$$\begin{aligned}
[f,g]_*(\vec{x}) &\equiv f * g\,(\vec{x}) - g * f\,(\vec{x}) = i\,\theta^{\mu\nu}\,\partial_\mu J_\nu^-(\vec{x}) \\
\{f,g\}_*(\vec{x}) &\equiv f * g\,(\vec{x}) + g * f\,(\vec{x}) = 2\,f(\vec{x})\,g(\vec{x}) + \theta^{\mu\nu}\,\partial_\mu J_\nu^+(\vec{x})
\end{aligned} \tag{A.3}$$

where the currents are given by

$$\begin{aligned}
J_\mu^-(\vec{x}) &= \sum_{m\geq 0}\frac{(-)^m}{2^{2m+1}\,(2m+1)!}\,\theta^{\mu_1\nu_1}\ldots\theta^{\mu_{2m}\nu_{2m}}\,\partial_{\mu_1}\ldots\partial_{\mu_{2m}}\,f(\vec{x})\,\partial_\mu\,\partial_{\nu_1}\ldots\partial_{\nu_{2m}}\,g(\vec{x}) \\
&- (f \leftrightarrow g) \\
J_\mu^+(\vec{x}) &= \sum_{m\geq 1}\frac{(-)^m}{2^{2m}\,(2m)!}\,\theta^{\mu_1\nu_1}\ldots\theta^{\mu_{2m-1}\nu_{2m-1}}\,\partial_{\mu_1}\ldots\partial_{\mu_{2m-1}}\,f(\vec{x})\,\partial_\mu\,\partial_{\nu_1}\ldots\partial_{\nu_{2m-1}}\,g(\vec{x}) \\
&+ (f \leftrightarrow g)
\end{aligned} \tag{A.4}$$

The fact that the Moyal bracket is a total derivative is a fundamental property allowing integration by parts and cyclic properties under integration to hold.

Let $\pi = \pi^a\,X_a \in \mathcal{G}$ where $\{X_a\}$ are some Lie algebra generators; then

$$[\pi_1,\pi_2]_* = \frac{1}{2}\,\{\pi_1^a,\pi_2^b\}_*\,[X_a,X_b] + \frac{1}{2i}\,[\pi_1^a,\pi_2^b]_*\,i\,\{X_a,X_b\} \tag{A.5}$$

that from properties (A.3), (A.4) closes in the algebra only for $\mathcal{G} = gl(N,C)$ or its restriction $u(N)$. The generators of $u(N)$ can be taken as those of $su(N)$ plus the identity, and in the paper is used the definition

$$\begin{aligned}
\kappa_{a_1\ldots a_n} &\equiv Tr\,(X_{a_1}\ldots X_{a_n}) \\
Tr\,(\ldots) &\equiv -tr_F\,(\ldots)
\end{aligned} \tag{A.6}$$

where the last line defines the scalar product adopted (denoted by "Tr") and "F" stands for the fundamental N-dimensional representation. In particular κ_{ab} is the metric used to rise and low indices in the algebra. However it is more convenient to expand in terms

[9] Compactifications are certainly possible, e.g. the NC torus and few other examples [2].

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of the generators of $gl(N, C)$, the matrices $(E_{ij})_{kl} = \delta_{ik}\,\delta_{jl}$. In this basis we must have in mind that $\pi^{ij\,*} = -\pi^{ji}$. By using this basis we easily get the various useful formulae

$$
\begin{aligned}
\kappa_{a_1}{}^{bc}\,\kappa_{a_2cb} &= \kappa_{a_1a_2b}{}^{b} = -N\,\kappa_{a_1a_2}\\
\kappa_{a_1}{}^{bc}\,\kappa_{a_2bc} &= \kappa_{a_1ba_2}{}^{b} = \kappa_{a_1}\,\kappa_{a_2}\\
\kappa_{a_1a_2a_3b}{}^{b} &= \kappa_{a_1}{}^{bc}\kappa_{a_2a_3cb} = \kappa_{a_1}{}^{bc}\kappa_{a_2db}\,\kappa_{a_3c}{}^{d} = -N\,\kappa_{a_1a_2a_3}\\
\kappa_{a_1a_2ba_3}{}^{b} &= \kappa_{a_1}{}^{bc}\,\kappa_{a_2ca_3b} = \kappa_{a_1}{}^{bc}\,\kappa_{a_2db}\,\kappa_{a_3}{}^{d}{}_{c} = \kappa_{a_1a_2}\,\kappa_{a_3}\\
\kappa^{b}{}_{ba_1...a_4} &= \kappa_{a_1a_2}{}^{cb}\,\kappa_{a_3a_4bc} = \kappa_{a_4bc}\,\kappa_{a_1a_2a_3}{}^{cb} = \kappa_{a_1a_2b}{}^{c}\,\kappa_{a_3}{}^{db}\,\kappa_{a_4cd}\\
&= \kappa_{a_1}{}^{be}\,\kappa_{a_2db}\,\kappa_{a_3}{}^{cd}\,\kappa_{a_4ec} = -N\,\kappa_{a_1...a_4}\\
\kappa^{b}{}_{a_1ba_2a_3a_4} &= \kappa_{a_1}{}^{b}{}_{a_2}{}^{c}\,\kappa_{a_3a_4bc} = \kappa_{a_1bc}\,\kappa_{a_2a_3a_4}{}^{bc} = \kappa_{a_4bc}\,\kappa_{a_2a_3}{}^{c}{}_{a_1}{}^{b} = \kappa_{a_2a_3b}{}^{c}\,\kappa_{a_1}{}^{bd}\,\kappa_{a_4cd}\\
&= \kappa_{a_2a_3b}{}^{c}\,\kappa_{a_4}{}^{db}\,\kappa_{a_1dc} = \kappa_{a_1ba_2c}\,\kappa_{a_4}{}^{bd}\,\kappa_{a_3dc} = \kappa_{a_4ba_1c}\,\kappa_{a_2}{}^{db}\,\kappa_{a_3cd}\\
&= \kappa_{a_1}{}^{be}\,\kappa_{a_2bc}\,\kappa_{a_4}{}^{cd}\,\kappa_{a_3de} = \kappa_{a_1}\,\kappa_{a_2a_3a_4}\\
\kappa^{b}{}_{a_1a_2ba_3a_4} &= \kappa_{a_1a_2}{}^{bc}\,\kappa_{a_3a_4bc} = \kappa_{a_1}{}^{b}{}_{a_4}{}^{c}\,\kappa_{a_3ba_2c} = \kappa_{a_4}{}^{bc}\,\kappa_{a_1a_2ba_3c} = \kappa_{a_1a_2b}{}^{c}\,\kappa_{a_3}{}^{bd}\,\kappa_{a_4dc}\\
&= \kappa_{a_1ba_4c}\,\kappa_{a_3}{}^{bd}\,\kappa_{a_2cd} = \kappa_{a_1ba_3c}\,\kappa_{a_2}{}^{db}\,\kappa_{a_4dc} = \kappa_{a_1}{}^{be}\,\kappa_{a_4bc}\,\kappa_{a_3}{}^{cd}\,\kappa_{a_2ed}\\
&= \kappa_{a_1}{}^{be}\,\kappa_{a_3bc}\,\kappa_{a_2}{}^{dc}\,\kappa_{a_4de} = \kappa_{a_1a_2}\,\kappa_{a_3a_4}
\end{aligned}
\tag{A.7}
$$

In terms of $g \equiv \exp(\lambda\pi)$ the NC left and right invariant Maurer-Cartan $u(N)$-valued forms have the expansions

$$
\begin{aligned}
L(g) &\equiv g^{-1}\,dg = \sum_{m\geq 1} \frac{\lambda^{m}}{m!}\,\omega_m(\pi) \qquad,\quad dL(g) + L(g)\wedge L(g) = 0\\
R(g) &\equiv dg\,g^{-1} = -\sum_{m\geq 1} \frac{(-\lambda)^{m}}{m!}\,\omega_m(\pi) \quad,\quad dR(g) - R(g)\wedge R(g) = 0\\
\omega_m(\pi) &= \sum_{p=1}^{m}(-)^{p+1}\binom{m}{p}\sum_{k=1}^{p}\pi^{k-1}\,d\pi\,\pi^{m-k} = [\dots[[d\pi,\overbrace{\pi]_*,\pi]_*,\dots,\pi]_*}^{m-1}
\end{aligned}
\tag{A.8}
$$

The even vertices of the action (1.1) come from I_0 while the odd ones come from the WZ term; however it is better to work out them in a unified way following [10]. To this end we consider

$$
\begin{aligned}
S[e^{\lambda\pi}] &= \int_0^{\lambda}dt\,\frac{dS[e^{t\pi}]}{dt} = \int_0^{\lambda}dt\,\lim_{\Delta t\to 0}\frac{1}{\Delta t}\,\left(S[g+\delta g] - S[g]\right)|_{\delta g = \Delta t\,g\,\pi\,,\,g=e^{t\pi}}\\
&= \frac{1}{\lambda^2}\int_0^{\lambda}dt\,\int d^2\vec{x}\,P_+^{ij}\,Tr\left(\partial_i\pi\,L_j(e^{t\pi})\right)
\end{aligned}
\tag{A.9}
$$

where in the last line we used (1.4). From here we read the vertices in the form

$$
S^n[\pi] = \frac{\lambda^{n-2}}{n!}\int d^2\vec{x}\,P_+^{ij}\,Tr\left(\partial_i\pi\,\omega_{n-1,j}(\pi)\right)\quad,\quad n\geq 3
\tag{A.10}
$$

B   Computation of $f(p;\theta)$

In this appendix we describe the exact computation of a one-loop integral; others integrals in the paper, more or less involved they be, are computed following similar same steps. We

would like to point out that in the context of dimensional regularization the introduction of Schwinger-Feynman parametrizations is the key to evaluate integrals; however in our regularization procedure they are not so useful due to the presence of the exponential factors in the propagators, so we must follow another route.

The function introduced in Section 3 is given by [10]

$$f(p;\theta) = \int \frac{d^2\vec{k}}{(2\pi)^2} \frac{\exp\left(-\frac{1}{2\Lambda^2}\left(\vec{k}^2 + (\vec{k}-\vec{p})^2\right) + i\,\theta\,\vec{p}\times\vec{k}\right)}{(\vec{k}^2 + m^2)\left((\vec{k}-\vec{p})^2 + m^2\right)} (\vec{p}\times\vec{k})^2 \tag{B.1}$$

It is clear almost by definition that is null at $p = 0$; then in the computation and by using rotational invariance we will take $\vec{p} = p\,\check{n}$ with $p > 0$, $\check{n} = (1,0)$, and the final result will take into account the mentioned fact. It will be convenient in what follows to introduce the dimensionless parameters (do not confound μ with any free scale)

$$\begin{array}{rclcrcl}
\sigma_\pm & \equiv & \sigma \pm 1 & , & \sigma & \equiv & \theta\,\Lambda^2 \\
x_0 & \equiv & \frac{1}{2}\left(\mu^2 + \sigma_+\,\sigma_-\right) & , & \mu & \equiv & \frac{2m}{p} \\
x_\pm & \equiv & \sigma^2 \pm \sqrt{\sigma^2 - \mu^2} & , & x_m & \equiv & \mu\,\sqrt{\sigma_+\,\sigma_-}
\end{array} \tag{B.2}$$

By making the change of variables $\vec{x} = \frac{2}{p}\vec{k} - \check{n}$ we write (B.1) as

$$f(p;\theta) = e^{-\frac{p^2}{4\Lambda^2}(1+\sigma^2)} \frac{p^2}{2\pi} \int \frac{d^2\vec{x}}{2\pi} \frac{\exp\left(-\frac{p^2}{4\Lambda^2}(\vec{x} + i\,\sigma\epsilon\check{n})^2\right)(\check{n}\times\vec{x})^2}{\left((\vec{x}+\check{n})^2 + \mu^2\right)\left((\vec{x}-\check{n})^2 + \mu^2\right)} \tag{B.3}$$

Now we must be careful because if the shift in a real vector just made is allowed, it is not so in general (and here in particular). In fact the integrand has poles in the complex x^2-plane and we must take them into account. Explicitly we can write

$$f(p;\theta) = -\frac{e^{\frac{m^2}{\Lambda^2} - \frac{p^2}{2\Lambda^2}}}{8\pi\,\sigma_+\,\sigma_-} p^2\,(f_s(p;\theta) + f_r(p;\theta)) \tag{B.4}$$

where $f_s(p;\theta)$ is the shifted integral

$$f_s(p;\theta) = \sigma_+\,\sigma_- \int \frac{d^2\vec{x}}{2\pi} \frac{e^{-\frac{p^2}{4\Lambda^2}(\vec{x}^2 + 2\,x_0)}(2\,i\,\check{n}\times\vec{x} - 2\,\sigma)^2}{\left((\vec{x} - i\,\sigma\epsilon\check{n} + \check{n})^2 + \mu^2\right)\left((\vec{x} - i\,\sigma\epsilon\check{n} - \check{n})^2 + \mu^2\right)} \tag{B.5}$$

and the contribution from the residues of the poles included in the strip $S = \{w = w_1 + iw_2 \ : \ w_1 \in \Re, w_2 \in [0,\sigma]\}$ is

$$\begin{aligned}
f_r(p;\theta) &= \int_{-\infty}^{\infty} dy \sum_{w_j \in S} Res(F(w;y); w_j) \\
F(w;y) &= -4\,\sigma_+\,\sigma_- \frac{i\,w^2\,\exp\left(-\frac{p^2}{4\Lambda^2}(w^2 - 2\,i\,\sigma\,w + y^2 + \mu^2 - 1)\right)}{(w^2 + (y+1)^2 + \mu^2)(w^2 + (y-1)^2 + \mu^2)}
\end{aligned} \tag{B.6}$$

---

[10] Dependence in the cut - off parameters are systematically omitted.

In what f_s concerns we introduce polar coordinates for $\vec{x} = (x^1, x^2)$ in the way

$$x^1 + i\, x^2 = z\, \sqrt{2\,(x - x_0)} \quad , \quad x \in [x_0, \infty) \quad , \quad |z|^2 = 1 \tag{B.7}$$

Then after a re-scaling in z we write

$$f_s(p; \theta) = \int_{x_0}^{\infty} dx\, e^{-\frac{p^2}{2\Lambda^2}\, x}\, I_s(x)$$

$$I_s(x) = \oint_{|z|^2 = 2(x - x_0)} \frac{dz}{2\pi i}\, \frac{(z^2 - 2\,\sigma\, z - 2\,x + 2\,x_0)^2}{z\,(z - z_{++}(x))\,(z - z_{-+}(x))\,(z - z_{+-}(x))\,(z - z_{--}(x))} \tag{B.8}$$

The contour integral $I_s(x)$ is then computed by using Cauchy theorem by evaluating the residues of the integrand in $\{z_0 = 0,\ z_{\pm+}(x) = -\frac{a_\mp(x)}{\sigma_+},\ z_{\pm-}(x) = -\frac{a_\pm(x)}{\sigma_-}\}$ where

$$a_\pm(x) = x - \sigma_+\,\sigma_- \pm \sqrt{x^2 - x_m{}^2} \tag{B.9}$$

All these poles lie on the real line and their residues are

$$\begin{aligned}
Res(z_0) &= 1 \\
Res(z_{\pm+}(x)) &= -Res(z_{\pm-}(x)) = \mp \frac{1}{2}\, \frac{(\sigma_+\, a_\pm(x) + \sigma_-\, a_\mp(x) + 2\,\sigma\,\sigma_+\,\sigma_-)^2}{(a_+(x) - a_-(x))\,(\sigma_+\, a_\pm(x) - \sigma_-\, a_\mp(x))} \\
&= \mp \frac{1}{2}\, \frac{(\sigma\, x \pm \sqrt{x^2 - x_m{}^2})^2}{\sqrt{x^2 - x_m{}^2}\,\left(x - \sigma_+\,\sigma_- \pm \sigma\,\sqrt{x^2 - x_m{}^2}\right)} \tag{B.10}
\end{aligned}$$

Of particular relevance will be the sum

$$\begin{aligned}
R(x) &\equiv Res(z_{++}(x)) + Res(z_{--}(x)) \\
&= \left(x^2 - \mu^2\,\sigma_+\,\sigma_-\right)^{-\frac{1}{2}}\left(-x - \sigma_+\,\sigma_- + \frac{\sigma_+\,\sigma_-}{2}\, \frac{\mu^2 + r_+}{x + r_+} + \frac{\sigma_+\,\sigma_-}{2}\, \frac{\mu^2 + r_-}{x + r_-}\right) \\
r_\pm &\equiv 1 \pm \sigma\,\sqrt{1 + \mu^2} \tag{B.11}
\end{aligned}$$

The next step is to analyze the contributions of theses residues. From it we get

$$f_s(p; \theta) = \begin{cases}
\int_{x_0}^{\infty} dx\, e^{-\frac{p^2}{2\Lambda^2}\, x}\, (1 + Res(z_{++}(x)) + Res(z_{--}(x))) & , \qquad \sigma^2 - \mu^2 < 0 \\[4pt]
\int_{x_0}^{x_-} dx\, e^{-\frac{p^2}{2\Lambda^2}\, x}\, (1 + Res(z_{++}(x)) + Res(z_{--}(x))) & \\
+ \int_{x_-}^{x_+} dx\, e^{-\frac{p^2}{2\Lambda^2}\, x}\, (1 + Res(z_{++}(x)) + Res(z_{-+}(x))) & \\
+ \int_{x_+}^{\infty} dx\, e^{-\frac{p^2}{2\Lambda^2}\, x}\, (1 + Res(z_{++}(x)) + Res(z_{--}(x))) & , \quad 0 < \sigma^2 - \mu^2 < 1 \\[4pt]
\int_{x_0}^{x_-} dx\, e^{-\frac{p^2}{2\Lambda^2}\, x}\, (1 + Res(z_{+-}(x)) + Res(z_{-+}(x))) & \\
+ \int_{x_-}^{x_+} dx\, e^{-\frac{p^2}{2\Lambda^2}\, x}\, (1 + Res(z_{++}(x)) + Res(z_{-+}(x))) & \\
+ \int_{x_+}^{\infty} dx\, e^{-\frac{p^2}{2\Lambda^2}\, x}\, (1 + Res(z_{++}(x)) + Res(z_{--}(x))) & , \qquad \sigma^2 - \mu^2 > 1
\end{cases} \tag{B.12}$$

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In what f_r concerns, the poles of $F(w; y)$ are localized in $w_{\pm+} = \pm i \sqrt{(y+1)^2 + \mu^2}$ and $w_{\pm-} = \pm i \sqrt{(y-1)^2 + \mu^2}$. Clearly only w_{++} and w_{+-} can lie inside the strip and they will contribute iff $|w_{+\pm}|^2 < \sigma^2$. The result of this analysis yields

$$
f_r(p; \theta) = \begin{cases} 0 & , \quad \sigma^2 - \mu^2 < 0 \\ \sigma_+ \, \sigma_- \; \int_{-\sqrt{\sigma^2-\mu^2}}^{\sqrt{\sigma^2-\mu^2}} dy \; \frac{\sqrt{y^2+\mu^2}}{y+1} \, e^{-\frac{p^2}{2\Lambda^2}\left(y+\sigma\sqrt{y^2+\mu^2}\right)} & , \quad \sigma^2 - \mu^2 > 0 \end{cases} \tag{B.13}
$$

We then make in the last case the change of variable

$$
x = y + \sigma \, \sqrt{y^2 + \mu^2} \tag{B.14}
$$

being careful with the inverse $y(x)$ to be considered that depends on the range of the parameters. We get

$$
f_r(p; \theta) = \begin{cases} 0 & , \quad \sigma^2 - \mu^2 < 0 \\ \int_{x_-}^{x_+} dx \; e^{-\frac{p^2}{2\Lambda^2}x} \, (-2\,Res(z_{-+}(x))) & , \quad 0 < \sigma^2 - \mu^2 < 1 \\ \int_{x_m}^{x_-} dx \; e^{-\frac{p^2}{2\Lambda^2}x} \, (2\,Res(z_{++}(x)) - 2\,Res(z_{-+}(x)) & \\ \quad + \; \int_{x_-}^{x_+} dx \; e^{-\frac{p^2}{2\Lambda^2}x} \, (-2\,Res(z_{-+}(x))) & , \quad \sigma^2 - \mu^2 > 1 \end{cases} \tag{B.15}
$$

From (B.4), (B.12) and (B.15) we finally obtain

$$
f(p; \theta) = -\frac{e^{\frac{m^2}{\Lambda^2} - \frac{p^2}{2\Lambda^2}}}{8\pi \, \sigma_+ \, \sigma_-} \, p^2 \; \times \begin{cases} \int_{x_0}^{\infty} dx \; e^{-\frac{p^2}{2\Lambda^2}x} \, (1 + R(x)) & , \quad \sigma_+ \, \sigma_- < \mu^2 \\ \int_{x_m}^{x_0} dx \; e^{-\frac{p^2}{2\Lambda^2}x} \, (-1 + R(x)) & \\ \quad + \int_{x_m}^{\infty} dx \; e^{-\frac{p^2}{2\Lambda^2}x} \, (1 + R(x)) & , \quad \sigma_+ \, \sigma_- > \mu^2 \end{cases} \tag{B.16}
$$

The reader should note the different form it displays in the two regions of parameters, a kind of phase transition around the value $\theta_c = \frac{\sqrt{p^2 + 4\,m^2}}{\Lambda^2 \, p}$. In particular if we like its value for $\theta = 0$ we must use the first expression; in the massless limit we get

$$
\begin{aligned}
\lim_{m \to 0} f(p; 0) &= \frac{p^2}{8\pi} \left(-2\,Ei\left(-\frac{p^2}{2\Lambda^2}\right) + Ei\left(-\frac{p^2}{4\Lambda^2}\right) + e^{-\frac{3}{8}\frac{p^2}{\Lambda^2}} \frac{\sinh(\frac{p^2}{8\Lambda^2})}{\frac{p^2}{8\Lambda^2}} \right) \\
&\xrightarrow{\Lambda \to \infty} \frac{p^2}{8\pi} \left(-\ln\frac{p^2}{\Lambda^2} + 1 - \gamma \right)
\end{aligned} \tag{B.17}
$$

where $Ei(x)$ is the exponential-integral function and γ the Euler constant (see [22]). Note that for $p = 0$ is null as should from the remark at the beginning; furthermore it displays a logarithmic singularity.

On the other hand, if we like its value at $\theta \neq 0$ for large enough cut-off Λ we must consider the second expression; in the massless limit we get

$$
\lim_{m \to 0} f(p; \theta) = \frac{p^2}{8\pi} \left(e^{-\frac{\sigma^2+3}{8}\frac{p^2}{\Lambda^2}} \frac{\sinh\frac{\sigma_+\sigma_- p^2}{8\Lambda^2}}{\frac{\sigma_+\sigma_- p^2}{8\Lambda^2}} + e^{\frac{\theta p^2}{2}} \left(-Ei(-\frac{\sigma_+ p^2}{2\Lambda^2}) + \frac{1}{2}\,Ei(-\frac{\sigma_+^2 p^2}{4\Lambda^2}) \right) \right)
$$

$$+ \quad e^{-\frac{\theta \, p^2}{2}} \left(-Ei(-\frac{\sigma_- p^2}{2\Lambda^2}) + \frac{1}{2} \, Ei(-\frac{\sigma_-{}^2 p^2}{4\Lambda^2}) \right) \Big)$$

$$\xrightarrow{\Lambda \to \infty} -\frac{p^2}{8\,\pi} \left(e^{\frac{\theta \, p^2}{2}} \, Ei(-\frac{\theta \, p^2}{2}) + e^{-\frac{\theta \, p^2}{2}} \, Ei(\frac{\theta \, p^2}{2}) \right) \qquad \text{(B.18)}$$

getting a finite result instead. Some remarks are in order. The first one concerns the massless limit; it is possible to prove that the results just obtained coincide with the ones we had obtained by putting $m = 0$ since the beginning. And with respect to the limit $\Lambda \to \infty$, the computation at $\theta \neq 0$ without cut-off agrees with it also. The second one is connected with the trick used to carry out the computations. There is a particular case, maybe of little physical interest, in which the computations simplifies a lot and corresponds to the scaling of the non commutative parameter defined by $\theta \, \Lambda^2 = 1$. In this case (for simplicity we consider the massless limit) f is written as

$$f(p; \theta) = e^{-\frac{\theta p^2}{2}} \, \frac{p^2}{2\pi} \int_0^\infty dk \, k \, e^{-\theta \, p^2 \, k^2} \int_{-\pi}^\pi \frac{d\phi}{2\pi} \, \frac{e^{\theta \, p^2 \, k \, e^{i\phi}}}{k^2 + 1 - 2\,k \, \cos\phi} \, \sin^2 \phi \qquad \text{(B.19)}$$

Going to the complex variable $z = e^{i\phi}$ the integral is straightforward and we finally get

$$f(p; \theta) = \frac{p^2}{8\pi} \, e^{-\frac{\theta p^2}{2}} \left(1 - \frac{\gamma}{2} - \ln\sqrt{\theta \, p^2} - \frac{e^{\theta p^2}}{2} \, Ei\left(-\theta p^2\right) \right) \qquad \text{(B.20)}$$

This result can be obtained from (B.16) as a special case; the important thing to observe is that the simplification in (B.19) occurs because we remain with just z in the exponent, in this way we remain with a meromorphic function (outside ∞) and we can apply Cauchy theorem. In the general case we have z and $\frac{1}{z}$, and this last factor represents an essential singularity at the origin which does not allow the residues calculation. This is the reason because it is necessary to make the shift in the exponential which leads to (B.4) in order to eliminate any z dependence. We finally remark that this approach works in the presence of any rational function of $\sin\phi$ and $\cos\phi$, in particular to compute any one-loop graph.

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